SOLUTION TO (MAX,+)-LABELLING PROBLEMS ON THE BASE OF THEIR EQUIVALENT TRASFORMATIONS

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Introduction

Peculiar optimisation problems, which arise in structural recognition, are analysed. The problems consist in optimisation of functions of large number of discrete arguments. Peculiarity of the problems consists in that the functions under optimisation are presented as a sum of large number of terms, each of them depending only on small number of variables, e.g. only on two arguments. The problems of such type are known as $(\max,+)$-labelling problems. Though the set of all possible problems of such type forms an *NP*-complete class, several tractable subclasses of this class are known. Two of them are most popular in pattern recognition. They are so called acyclic and supermodular problem classes.

A concept of equivalent transformations of $(\max,+)$-problems is defined in the paper. On the base of such transformations an approach to solution to $(\max,+)$-problems is described. It is shown that the set of problems, which can be solved with their equivalent transformations, includes all acyclic and supermodular $(\max,+)$-problems.

1. Main concepts and a subject of research

It is known that several problems of image structural recognition can be reduced to specific discrete optimisation problems [1, 2, 3, 4, 5, 6, 7, 8, 9]. In spite of wide variety of their applied content they can be described uniformly in the certain form known as a $(\max,+)$-labelling problem. The problem formulation is based on the following concepts.

Let T be a set of objects, e.g. set of pixels in a vision field. Elements of the set will be denoted as t or t', $t \in T$, $t' \in T$. Let a subset $\Im \subset T \times T$ of object pairs be given, which defines a neighbourhood relation. Designation $tt' \in \Im$ will mean that the objects t and t' are neighbours. Designation $N(t)$ will be used for the set of objects, which are neighbouring to the object t. It is assumed here and below that the graph of neighbourhood relation $\Im$ is connected.

Let K be a set, whose elements will be referred to as labels. For each label k and each object t a number $q_t(k) \in R$ is defined. Similarly, for each pair $tt' \in \Im$ of neighbours and each pair of labels k and k' a number $g_{tt'}(k,k') \in R$ is defined. These numbers will be referred to as (local) qualities or weights. The weights $q_t(k)$ and $g_{tt'}(k,k')$ are real numbers, none of them being $-\infty$. A labelling is defined as a function $\bar{k}: T \to K$, which assigns a label $k(t)$ to each object t. A set of all possible functions of such format will be denoted as K^T. A (global) quality $G(\bar{k})$ of a labelling $\bar{k}: T \to K$ is defined as a sum of its local qualities, i.e.

$$G(\bar{k}) = \sum_{tt' \in \Im} g_{tt'}(k(t), k(t')) + \sum_{t \in T} q_t(k(t)) . \quad (1)$$

A $(\max, +)$-labelling problem consists in calculating a quality

$$G = \max_{\bar{k} \in K^T} \left[\sum_{tt' \in \Im} g_{tt'}(k(t), k(t')) + \sum_{t \in T} q_t(k(t)) \right] \quad (2)$$

of the best labelling and/or looking for the best labelling

$$\bar{k}^* = \arg\max_{\bar{k} \in K^T} G(\bar{k}) . \quad (3)$$

One can assume without loss of generality that all numbers $q_t(k)$ are 0. Hereinafter the quality of a labelling will be defined either as (1) or as

$$G(\bar{k}) = \sum_{tt' \in \Im} g_{tt'}(k(t), k(t')) \quad (4)$$

depending on a convenience of such or other definition. The set of all possible problems of the type (1) or (3) forms an NP-complete class. However, three subclasses of this class are known to be tractable. We call them acyclic problems, supermodular problems and problems, which are equivalent to trivial ones.

We will say that a $(\max, +)$-problem is acyclic if a neighbourhood $\Im$ contains no cycle. It is well-known (see [10] for example) that such problems are tractable at each weights $q_t(k)$ and $g_{tt'}(k,k')$, a dynamic programming being a generally used tool for their solution.

It resulted is stated in a lot of new results in structural recognition of last decade [2, 6, 7, 9, 11, 12, 13, 14] that so-called supermodular $(\max, +)$-problems are also tractable. The supermodularity consists in the following restrictions on weights $g_{tt'}(k,k')$ and a set K of labels:

1. The set K of labels is an ordered set.

2. For each four labels k, l и k', l', such that $k < l$ and $k' < l'$, and for each pair $tt' \in \Im$ of neighbours the inequality

$$g_{tt'}(k,k') + g_{tt'}(l,l') \geq g_{tt'}(k,l') + g_{tt'}(l,k') \tag{5}$$

holds. Supermodular $(\max,+)$-problems are tractable at arbitrary weights $q_t(k)$ and arbitrary neighbourhood $\Im$. Their solution is based on the fact that they can be reduced to polinomially solvable problem of maximal flow in a graph.

The class of solvable problems, which are in a sense equivalent with trivial ones, is not so popular as acyclic and supermodular ones though these problems were formulated much earlier. The first papers on the problem were published at 70[th] of the last century [1, 15]. Later on the similar ideas were researched using another terminology (reparametrization, belief propagation etc.) [16, 17, 18]. The approach at whole is rather completely reviewed in the report [19]. The approach is based on the following concepts.

Let $(\max,+)$-problem be given with some weights $g_{tt'}(k,k')$ and zero weights $q_t(k)$. Let for each pair $tt' \in \Im$ of neighbours the label pairs with maximal local quality be chosen. It means that for each pair of labels k and k' a binary value $\overline{g}_{tt'}(k,k')$ is defined that indicates whether the label pair k and k' is chosen for the pair $tt' \in \Im$ of objects or not:

$$\overline{g}_{tt'}(k,k') = \begin{cases} 1, \text{ if } g_{tt'}(k,k') = \max\limits_{k^* \in K, k^{**} \in K} g_{tt'}(k^*,k^{**}) \\ 0, \text{ if } g_{tt'}(k,k') < \max\limits_{k^* \in K, k^{**} \in K} g_{tt'}(k^*,k^{**}). \end{cases} \tag{6}$$

The problem is called trivial if a labelling $\overline{k}^*$ exists, which is composed only with chosen label pairs. More precisely, the problem is trivial by definition if a labelling $\overline{k}^*$ exists such that

$$\mathop{\&}_{tt' \in \Im} \overline{g}_{tt'}(k^*(t), k^*(t')) = 1. \tag{7}$$

An equivalency of two $(\max,+)$-problems is defined in the following natural way. Let two $(\max,+)$-problems be given, first of them with weights $g^1{}_{tt'}(k,k')$ and second with weights $g^2{}_{tt'}(k,k')$. These two problems are called equivalent if the (global) quality of each labelling in the first problem equals the quality of the same labelling in the second problem. More precisely, two problems are equivalent if the equality

$$\sum_{tt' \in \Im} g^1{}_{tt'}(k(t),k(t')) = \sum_{tt' \in \Im} g^2{}_{tt'}(k(t),k(t')) \tag{8}$$

holds for each labelling $\bar{k} \in K^T$.

If a $(\max,+)$-problem is equivalent to trivial one then the quality (2) of the best labelling can be calculated in a polynomial time because the calculation can be performed with linear programming [1, 2, 20]. However, due to large number of variables and restrictions general linear programming algorithms (e.g. simplex-method) are inappropriate. All known algorithms, both the very first [15] and subsequent ones [16, 17, 18] try to transform an initial problem to a trivial one with special-purpose algorithms, more simple than general-purpose methods. Unfortunately, all these attempts suffer from common drawback. They do not guarantee that an initial problem will be transformed into trivial one even if such trivial equivalent exists. Later on we formulate this drawback more definitely. For the present we mention only that the known algorithms transform an initial problem into the problem, which satisfy only necessary conditions of triviality, not sufficient. So, the question remains to be unanswered, whether a practically good algorithm exists, which can trivialize any $(\max,+)$-problem, naturally, if for the given problem a required trivial equivalent exists at all. In this paper we try to fill this gap in our knowledge of $(\max,+)$-problem equivalency. At the same time, in this paper the method of equivalent transformation will be completely described, including the important fact that each supermodular problem is equivalent to trivial one, triviality being meant in an exact, above defined sense. The same will be shown for acyclic problems. Analysis of such properties of trivial and equivalent problems is dissipated in several publications, particularly, in some former article of the first author, as well as in the review [19]. This information is included in the present paper too for its autonomous completeness.

The paper is organised in the following way. An interrelation of the known $(\max,+)$-problem classes is considered in the Section 2. It is shown that all supermodular as well as acyclic problems are equivalent to some trivial problems. Equivalence classes of (max,+)-problems are analysed in the Section 3 and it is shown that transformations of the problems with so called potentials ([1]) exhaust all possible equivalent transformations. It is also shown that the transformation of $(\max,+)$-problem into trivial one is a problem of convex minimisation, but non-smooth one. An algorithm to solution this optimisation problem is described in the Section 4. The algorithm is based on the method of sub-gradient descent [21]. In the Section 5 obtained results are discussed and the main difference of proposed approach as compared with known ones is stated. Then, in the Section 6, it is shown how the idea of equivalent transformation of $(\max,+)$-problems has to be effectively used in a special case of neighbourhood, typical for two-dimensional field of vision. An algorithm that follows from these considerations is described in the Section 7. Description of various experiments of image labelling in the Section 8 concludes the paper.

2. Acyclic and supermodular $(\max,+)$-problems and problems equivalent to trivial ones

We will formulate and prove two theorems about equivalency of acyclic and supermodular problems to trivial one. The following lemma is required to prove the first theorem.

Lemma 1 *Let for some fixed set T, $|T| \geq 2$, and some fixed neighbourhood $\Im$ a $(\max,+)$-problem be equivalent to trivial one at arbitrary weights $g_{tt'}(k,k')$, $tt' \in \Im$, $k \in K$, $k' \in K$. Let t' be an object, which does not belong in T, and t'' be an object, which belongs to T. In this case, the $(\max,+)$-problem on the set $T' = T \cup \{t'\}$ and neighbourhood $\Im' = \Im \cup \{t't''\}$ is equivalent to trivial one at arbitrary weights $g'_{tt'}(k,k')$, $tt' \in \Im'$, $k \in K$, $k' \in K$.*

Proof. Let $t^* \in T$ be an arbitrary object, which is neighbouring to t'', $t^*t'' \in \Im$. Let for each label $k'' \in K$ and for arbitrary numbers $\varphi(k'')$ the following transformation of weights be defined,

$$\begin{aligned} \bar{g}_{t''t'}(k'',k') &= g_{t''t'}(k'',k') + \varphi(k''), \qquad k' \in K, \\ \bar{g}_{t''t^*}(k'',k^*) &= g_{t''t^*}(k'',k^*) - \varphi(k''), \quad k^* \in K. \end{aligned} \tag{9}$$

This transformation is equivalent because it changes a (global) quality of no labelling. Really, the quality of labelling $\bar{k} \in K^T$ contains the terms $g_{t''t'}(k(t''),k(t'))$ and $g_{t''t^*}(k(t''),k(t^*))$ before transformation and the terms $\bar{g}_{t''t'}(k(t''),k(t'))$ and $\bar{g}_{t''t^*}(k(t''),k(t^*))$ after transformation. The first pair of the weights differs from the second one, however, due to (9) their sums are equal,

$$\bar{g}_{t''t'}(k(t''),k(t')) + \bar{g}_{t''t^*}(k(t''),k(t^*)) = g_{t''t'}(k(t''),k(t')) + g_{t''t^*}(k(t''),k(t^*)).$$

All other terms in the quality of this labelling remains unchanged.

Let us fulfil the transformation (9) in such way that each number $\bar{g}_{t''t'}(k'',k')$, $k'' \in K, k' \in K$, becomes non-positive and for each label k'' the number $\bar{g}_{t''t'}(k'',k')$, becomes 0 at least for one label $k' \in K$.

According to lemma statement the $(\max,+)$-problem for the set T, neighbourhood $\Im$ and weights $\bar{g}_{tt'}(k,k')$ is equivalent to trivial one. It means that there exist such weights $\tilde{g}_{tt'}(k,k')$ and such labelling $\bar{k}^0 : T \to K$ that equality $\tilde{g}_{tt'}(k^0(t),k^0(t')) = \max\limits_{k \in K, k' \in K} \tilde{g}_{tt'}(k,k')$ is valid for each pair

tt' of objects, which are neighbours in $\Im$. Let us extend this labelling onto set $T' = T \cup \{t'\}$ so that for object t' such label $k^0(t')$ will be assigned that $\bar{g}_{t''t'}(k^0(t''), k^0(t')) = 0$. For the labelling $\bar{k}^0 : T' \to K$ the equality $\tilde{g}_{tt'}(k^0(t), k^0(t')) = \max_{k \in K, k' \in K} \tilde{g}_{tt'}(k, k')$ holds for each pair of objects, which are neighbours in $\Im'$. So, for the object set T', neighbourhood $\Im'$ and weights $\tilde{g}_{tt'}(k, k')$, $tt' \in \Im$, and $\bar{g}_{t''t'}(k'', k')$ the $(\max, +)$-problem is trivial. It is equivalent to an initial one.

■

Theorem 1. *Each $(\max, +)$-problem on a set T with acyclic neighbourhood $\Im$ has a trivial equivalent.*

Proof. Let us define a sequence $\{T_i, \Im_i\}$, $i = 1, \ldots, n-1$, of graphs in such way that the set T_1 consists of two objects, which are neighbouring with each other, the graph $\{T_{n-1}, \Im_{n-1}\}$ is the graph $\{T, \Im\}$ and each two graphs $\{T_i, \Im_i\}$ and $\{T_{i+1}, \Im_{i+1}\}$ in the sequence satisfy the conditions of lemma 1 about relation between graphs $\{T, \Im\}$ and $\{T', \Im'\}$. A $(\max, +)$-problem for the graph $\{T_1, \Im_1\}$ is evidently trivial because T_1 consists of only two objects. For each $i = 1, \ldots, n-2$ the graphs $\{T_i, \Im_i\}$ and $\{T_{i+1}, \Im_{i+1}\}$ satisfy the statement of lemma 1. So, $(\max, +)$-problem for each graph $\{T_i, \Im_i\}_{i=1}^{n-1}$ is equivalent to trivial one, as well as the initial problem because it coincides with the problem for graph $\{T_{n-1}, \Im_{n-1}\}$. ■

In fact, the proof of the lemma 1 and the theorem 1 shows the way for developing algorithm that transforms any acyclic $(\max, +)$-problem into trivial one and so solves any acyclic $(\max, +)$-problem. Such algorithm will not differ from the well-known solution to problems of such type with dynamic programming (see, for example, [10]). However, the main idea of such solution with dynamic programming cannot be generalised for arbitrary neighbourhood, not obligatory acyclic. As to equivalent transformations of $(\max, +)$-problems, they are defined for $(\max, +)$-problems with arbitrary neighbourhood and so, hopefully, can show the way for solution to not only acyclic problems. We will show how this possibility can be realised. The dominant role in equivalent transformation of a problem into trivial one is performed by the problem power, which is defined as a number

$$E = \sum_{tt' \in \Im} \max_{k \in K, k' \in K} g_{tt'}(k,k') .$$

(10)

It is evident from the definitions (10) and (1) that for each labelling $\bar{k} \in K^T$ the inequality

$$G(\bar{k}) \le E \tag{11}$$

holds. The following lemma is also almost evident.

Lemma 2. *A* $(\max,+)$*-problem is trivial if and only if a labelling* $\bar{k}^*$ *exists, for which the equality* $G(\bar{k}^*) = E$ holds.

Though the next lemma is slightly less evident it can be easily proved.

Lemma 3. *Let* Z *be an equivalence problem class that contains at least one trivial problem* $z^* \in Z$. *In this case:*

1) the problem z^* *minimises problem power in the class* Z.

2) each problem $z^{**} \in Z$, *which minimises problem power in the class* Z, *is trivial.*

Proof. Due to lemma 2 such labelling $\bar{k}$ exists, for which the equality $G(\bar{k}, z^*) = E(z^*)$ holds, where $G(\bar{k}, z^*)$ is a quality of the labelling $\bar{k}$ in the problem z^* and $E(z^*)$ is a power of this problem. Let z' be any problem from the class Z. Since the problems z^* and z' are equivalent, the equality $G(\bar{k}, z^*) = G(\bar{k}, z')$ holds. On the other hand, for the labelling $\bar{k}$, according to (10), inequality $G(\bar{k}, z') \le E(z')$ holds. What was said can be represented with a chain

$$E(z^*) = G(\bar{k}, z^*) = G(\bar{k}, z') \le E(z')$$

and so the first statement of the lemma is proved.

Let z^{**} be some problem from the class Z with minimal in this class power. It means that $E(z^{**}) = E(z^*)$. By the lemma statement, the problem z^* is trivial and so, due to lemma 2, $G(\bar{k}, z^*) = E(z^*)$. The problems z^* and z^{**} are equivalent that means $G(\bar{k}, z^*) = G(\bar{k}, z^{**})$. The last three equalities result in equality $E(z^{**}) = G(\bar{k}, z^{**})$. It means due to lemma 2 that the problem z^{**} is trivial and so the second statement of the lemma is proved. ■

Lemma 4. *Let* z *be some* $(\max,+)$*-problem and* Z *be class of problems, which are equivalent to* z. *In this case class* Z *contains a problem, which minimises power in this class.*

Proof. A power $E=\sum_{tt'\in\Im}\max_{k\in K,k'\in K} g_{tt'}(k,k')$ can be represented in a form

$$E=\min\sum_{tt'\in\Im} h(t,t'),$$

where $h(t,t')$ are auxiliary variables, which satisfy the inequalities

$$h(t,t')\geq g_{tt'}(k,k'),\quad tt'\in\Im,\quad k\in K,\quad k'\in K\,.$$

Equivalence of two problems is defined by the system of linear equalities

$$\sum_{tt'\in\Im}\overline{g}_{tt'}(k(t),k(t'))=\sum_{tt'\in\Im} g_{tt'}(k(t),k(t')),\quad \overline{k}\in K^T\,.$$

Looking for $(\max,+)$-problem, which is equivalent to z and has the minimal power in Z, is the following problem of linear programming:

$$\begin{aligned}&\min\sum_{tt'\in\Im} h(t,t')\\ &h(t,t')\geq \overline{g}_{tt'}(k,k'),\quad tt'\in\Im,\quad k\in K,\quad k'\in K,\\ &\sum_{tt'\in\Im}\overline{g}_{tt'}(k(t),k(t'))=\sum_{tt'\in\Im} g_{tt'}(k(t),k(t')),\quad \overline{k}\in K^T.\end{aligned}$$

Numbers $\overline{g}_{tt'}(k,k')$ and $h(t,t')$ are variables in the problem that have to satisfy the certain restrictions. Evidently, these restrictions are not contradictory, because they are fulfilled at least at $\overline{g}_{tt'}(k,k')=g_{tt'}(k,k')$. The function under minimisation is bounded from below, because it cannot be less than a quality of some labelling $\overline{k}$. Due to the known theorems of linear optimisation [22] for such problems the set of allowed solutions (in our case, the set Z of the problem equivalent to z) contains the solution with minimal value of criterion function (in our case, the problem with minimal power). ■

Lemma 5. *Let z be a supermodular $(\max,+)$-problem defined for a set T, neighbourhood $\Im$ and weights $g_{tt'}(k,k')$, $tt'\in\Im$, $k\in K$, $k'\in K$. Let $\varphi_{tt'}(k)$, $tt'\in\Im$, $k\in K$, $k'\in K$, be arbitrary real numbers. Let $\overline{z}$ be a $(\max,+)$-problem defined for the same set T, same neighbourhood $\Im$, but for weights $\overline{g}_{tt'}(k,k')=g_{tt'}(k,k')+\varphi_{tt'}(k)+\varphi_{t't}(k')$. In this case the problem $\overline{z}$ is supermodular too.*

Proof. For the problem z due to its supermodularity the inequality

$$g_{tt'}(l,l')+g_{tt'}(m,m')\geq g_{tt'}(l,m')+g_{tt'}(m,l')$$

holds for each $tt' \in \Im$ and each labels $l < m$ и $l' < m'$. It results immediately that

$$g_{tt'}(l,l') + \varphi_{tt'}(l) + \varphi_{t't}(l') + g_{tt'}(m,m') + \varphi_{tt'}(m) + \varphi_{t't}(m') \geq \\ g_{tt'}(l,m') + \varphi_{tt'}(l) + \varphi_{t't}(m') + g_{tt'}(m,l') + \varphi_{tt'}(m) + \varphi_{t't}(l').$$

The left part in this inequality is $\bar{g}_{tt'}(l,l') + \bar{g}_{tt'}(m,m')$ and right part is $\bar{g}_{tt'}(l,m') + \bar{g}_{tt'}(m,l')$. It means that $\bar{g}_{tt'}(l,l') + \bar{g}_{tt'}(m,m') \geq \bar{g}_{tt'}(l,m') + \bar{g}_{tt'}(m,l')$ and supermodularity of the problem $\bar{z}$ is proved. ■

Theorem 2. *Let z be a supermodular $(\max,+)$-problem defined for a set T, a neighbourhood $\Im$ and weights $g_{tt'}(k,k')$, $tt' \in \Im$, $k \in K$, $k' \in K$. There exists a trivial problem z^*, which is equivalent to z. Namely, z^* is a problem with minimal power over the problems, which are equivalent to z.*

Proof. We will prove that each supermodular problem z either is trivial or can be equivalently transformed into the supermodular problem with lower power. So the theorem will be proved. Really, for the supermodular problem z^* with minimal power (an existence of such problem is proved by lemma 4) no problem with lower power is possible. Consequently, the problem z^* can be only trivial.

We will show with the following algorithm that either the supermodular problem is trivial or it is possible to decrease its power. Input data for the algorithm are given by numbers $g_{tt'}(k,k')$, $tt' \in \Im$, $k \in K$, $k' \in K$.

1. Create the set

$$K_{tt'} = \left\{ (k,k') \middle| g_{tt'}(k,k') = \max_{l,l'} g_{tt'}(l,l') \right\}$$

for each pair $tt' \in \Im$ of neighbours.

2. For each pair $tt' \in \Im$ of objects define such labels $k_{tt'}(t)$ and $k_{tt'}(t')$ that for each label pair $(k,k') \in K_{tt'}$ the inequalities $k_{tt'}(t) \geq k$, $k_{tt'}(t) \geq k'$ are valid. In other words, the label $k_{tt'}(t)$ is the highest label in the set $\left\{ k \middle| \exists k' : (k,k') \in K_{tt'} \right\}$, and $k_{tt'}(t')$ is the highest label in the set $\left\{ k' \middle| \exists k : (k,k') \in K_{tt'} \right\}$. Let us show that the set $K_{tt'}$ contains the label pair $(k_{tt'}(t), k_{tt'}(t'))$ defined in mentioned way. Really, there exists such label l' that $l' \leq k_{tt'}(t')$ and $(k_{tt'}(t), l') \in K_{tt'}$ as well as the label l exists such that $l \leq k_{tt'}(t)$ and $(l, k_{tt'}(t')) \in K_{tt'}$. It means that

$$g_{tt'}\left(k_{tt'}\left(t\right),l'\right)=g_{tt'}\left(l,k_{tt'}\left(t'\right)\right)=\max_{s,s'}g_{tt'}\left(s,s'\right)$$

and, due to supermodularity of the problem,

$$g_{tt'}\left(k_{tt'}\left(t\right),k_{tt'}\left(t'\right)\right)=\max_{s,s'}g_{tt'}\left(s,s'\right).$$

This, in turn, means that the set $K_{tt'}$ contains above defined label pair $\left(k_{tt'}\left(t\right),k_{tt'}\left(t'\right)\right)$.

3. Find such triple $t,\ t',\ t''$ of objects that $tt'\in\Im$, $tt''\in\Im$ and $k_{tt'}\left(t\right)\neq k_{tt''}\left(t\right)$.

If such triple does not exist stop the algorithm because the current problem is trivial.

4. Assume for unambiguity that $k_{tt'}\left(t\right)>k_{tt''}\left(t\right)$ and fulfil the transformation

$$\begin{aligned}&g_{tt'}(k_{tt'}\left(t\right),k'):=g_{tt'}(k_{tt'}\left(t\right),k')-\varphi,\quad k'\in K,\\&g_{tt''}(k_{tt'}\left(t\right),k''):=g_{tt''}(k_{tt'}\left(t\right),k'')+\varphi,\ \ k''\in K.\end{aligned}\qquad(12)$$

Due to lemma 5 this transformation does not violate the problem's supermodularity. It is also easy to see that transformation (12) changes a quality of none labelling, i.e. is an equivalent transformation.

At positive values of φ the transformation (12) decreases the values $g_{tt'}\left(k_{tt'}\left(t\right),k'\right)$, $k'\in K$. Consequently, some label pairs have been excluded from the set $K_{tt'}$. At the same time the transformation (12) increases numbers $g_{tt''}\left(k_{tt''}\left(t\right),k''\right)$, $k''\in K$. However, none of the pairs $\left(k_{tt''}\left(t\right),k''\right)$, $k''\in K$, is contained in the set $K_{tt''}$ and so at sufficiently small but positive value of φ the set $K_{tt''}$ remains unchanged.

5. If the set $K_{tt'}$ occurs to be empty stop the algorithm because the initial problem is equivalently transformed into problem with lower power.

6. Go to 2. ■

3. Constructive description of classes of equivalent problems

Several transformations of $(\max,+)$-problems, which were evidently equivalent, were used in the previous section to prove that two widely known classes of $(\max,+)$-problems are equivalent to trivial ones. The next aim is to construct tools for equivalent transformation of arbitrary $(\max,+)$-problem and its transformation into trivial one, certainly, if for the given problem such trivial equivalent exists at all. For this aim it is necessary to describe completely whole set of equivalent transformation of $(\max,+)$-problems.

Let us define auxiliary concept of zero $(\max,+)$-problem as a problem, for which a quality of each labelling $\bar{k} \in K^T$ is 0. The following evident lemma shows how the complete description of a set of zero problems describes exhaustively all equivalence classes of $(\max,+)$-problems.

Lemma 6. *Two* $(\max,+)$*-problems with weights* $g^1_{tt'}(k,k'), q^1_t(k)$ *and* $g^2_{tt'}(k,k'), q^2_t(k)$ *are equivalent if and only if a problem with weights* $g_{tt'}(k,k') = g^1_{tt'}(k,k') - g^2_{tt'}(k,k')$ *and* $q_t(k) = q^1_t(k) - q^2_t(k)$ *is a zero problem.* ■

Let some number $\varphi_{tt'}(k)$ will be defined for each $tt' \in \Im$ and each $k \in K$. The number $\varphi_{tt'}(k)$ will be referred to as a potential that the label k of object t radiates towards to neighbour t', or simply potential.

Theorem 3. A $(\max,+)$*-problem with weights* $g_{tt'}(k,k')$ *and* $q_t(k)$ *is a zero problem if and only if such potentials* $\varphi_{tt'}(k)$, $tt' \in \Im$, $k \in K$, *and numbers* $h(t)$, $t \in T$, *exist that satisfy the equations*

$$\begin{cases} g_{tt'}(k,k') = \varphi_{tt'}(k) + \varphi_{t't}(k'), \quad tt' \in \Im, \quad k \in K, \quad k' \in K, \\ q_t(k) = -\sum\limits_{t' \in N(t)} \varphi_{tt'}(k) + h(t), \quad t \in T, \quad k \in K, \\ \sum\limits_{tt' \in \Im} h(t) = 0. \end{cases} \tag{13}$$

Proof. Let for the given weights $g_{tt'}(k,k')$ the potentials $\varphi_{tt'}(k)$ and numbers $h(t)$ exist, which satisfy linear restrictions (13). A quality of an arbitrary labelling $\bar{k} \in K^T$ equals

$$\sum_{tt' \in \Im} g_{tt'}(k(t),k(t')) + \sum_{t \in T} q_t(k(t)) =$$

$$\sum_{tt' \in \Im} \big(\varphi_{tt'}(k(t)) + \varphi_{t't}(k(t'))\big) + \sum_{t \in T} \left(h(t) - \sum_{t' \in N(t)} \varphi_{tt'}(k(t)) \right).$$

Each potential $\varphi_{tt'}(k)$ appears in this expression either twice or never. Moreover, if some potential $\varphi_{tt'}(k(t))$ enters into the sum $\sum\limits_{tt' \in \Im} \big(\varphi_{tt'}(k(t)) + \varphi_{t't}(k(t'))\big)$ positively then the same potential enters into the sum $\sum_{t\in T}\left(h(t)-\sum_{t'\in N(t)}\varphi_{tt'}(k(t))\right)$ negatively. So the quality of each labelling is $\sum_{t\in T}h(t)$ and due to (13) is 0.

It is somewhat more laborious to prove that for each zero problem the potentials $\varphi_{tt'}(k)$ and numbers $h(t)$ exist, which satisfy the restrictions (13). This proof is fulfilled in the following four steps.

1. We will prove that if the weights $g_{tt'}(k,k')$ and $q_t(k)$ define zero problem then an equality

$$g_{tt'}(k_1,k_1')+g_{tt'}(k_2,k_2')=g_{tt'}(k_1,k_2')+g_{tt'}(k_2,k_1') \tag{14}$$

holds for each pair $tt'\in\Im$ of neighbours and for each four-tuple k_1,k_1',k_2,k_2' of labels. Functions with such property are known as modular ones.

Let us choose and fix for further consideration two neighbours t and t', two labels k_1 and k_2 as possible labels of the object t and two labels k_1' and k_2' as possible labels of the object t'. Let us choose arbitrary labelling of all other objects and fix it for further consideration. As a result four labellings $\bar{k}_1,\bar{k}_2,\bar{k}_3,\bar{k}_4$ will be obtained such that

$k_1(t)=k_1,\ k_1(t')=k_1',\ k_2(t)=k_2,\ k_2(t')=k_2',\ k_3(t)=k_1,\ k_3(t')=k_2',\ k_4(t)=k_2,\ k_4(t')=k_1'$.

By the theorem statement quality of each obtained labelling is 0. A quality of the first of them can be represented in a form

$$G(\bar{k}_1)=g_{tt'}(k_1,k_1')+f_t(k_1)+f_{t'}(k_1')+A=0\,, \tag{15}$$

where

$$f_t(k_1)=\sum_{\substack{t''\in N(t)\\ t''\neq t'}}g_{tt'}(k_1,k_1(t''))+q_t(k_1)\,,$$

$$f_{t'}(k_1')=\sum_{\substack{t''\in N(t')\\ t''\neq t}}g_{tt'}(k_1',k_1(t''))+q_{t'}(k_1')\,,$$

$$A=\sum_{\substack{t^*t^{**}\in\Im,\\ t^*\neq t,t^*\neq t',\\ t^{**}\neq t,t^{**}\neq t'}}g_{t^*t^{**}}(k_1(t^*),k_1(t^{**}))+\sum_{\substack{t^*\in T,\\ t^*\neq t,t^*\neq t'}}q_{t^*}(k_1(t^*))\,.$$

One can see that the term A will appear in qualities of all other three labellings, the term $f_t(k_1)$ will appear in a quality of the third labelling and the term $f_{t'}(k_1')$ will appear in a quality of the fourth labelling. In the same way as a quality (15) of the first labelling qualities of other three labellings are

$$G(\bar{k}_2) = g_{tt'}(k_2, k'_2) + f_t(k_2) + f_{t'}(k'_2) + A = 0\,, \tag{16}$$

$$G(\bar{k}_3) = g_{tt'}(k_1, k'_2) + f_t(k_1) + f_{t'}(k'_2) + A = 0\,, \tag{17}$$

$$G(\bar{k}_4) = g_{tt'}(k_2, k'_1) + f_t(k_2) + f_{t'}(k'_1) + A = 0\,. \tag{18}$$

As a result of summation (15) and (16) and subtracting (17) and (18) the equality

$$g_{tt'}(k_1, k'_1) + g_{tt'}(k_2, k'_2) - g_{tt'}(k_1, k'_2) - g_{tt'}(k_2, k'_1) = 0 \tag{19}$$

is obtained that proves (14).

2. Let us prove that for each modular function $g_{tt'} : K \times K \to R$ such potentials $\varphi_{tt'}(k)$ exist that

$$g_{tt'}(k, k') = \varphi_{tt'}(k) + \varphi_{t't}(k')\,.$$

The potentials have to be found in the following way. At first an arbitrary label k_0 is chosen and then the numbers $\varphi_{tt'}(k)$ and $\varphi_{t't}(k)$ are defined as

$\varphi_{tt'}(k_0) = 0\,,$

$\varphi_{t't}(k) = g_{tt'}(k_0, k)\,,\ k \in K\,,$

$\varphi_{tt'}(k) = g_{tt'}(k, k_0) - \varphi_{t't}(k_0)\,,\ k \in K\,.$

Evidently, for the defined numbers $\varphi_{tt'}(k)$ and $\varphi_{t't}(k)$ the equalities

$$g_{tt'}(k, k_0) = \varphi_{tt'}(k) + \varphi_{t't}(k_0)\,, \tag{20}$$

$$g_{tt'}(k_0, k) = \varphi_{tt'}(k_0) + \varphi_{t't}(k) \tag{21}$$

hold for each $k \in K$.

Let us show that for the defined numbers $\varphi_{tt'}(k)$ and $\varphi_{t't}(k)$ the equality $g_{tt'}(k, k') = \varphi_{tt'}(k) + \varphi_{t't}(k')$ also holds for each pair k, k'. Due to the modularity of the function $g_{tt'}$ the equality

$$g_{tt'}(k, k') = g_{tt'}(k, k_0) - g_{tt'}(k_0, k_0) + g_{tt'}(k_0, k')\,.$$

is valid. Let us perform the terms in right part according to (20) and (21) and obtain

$$g_{tt'}(k, k') = \varphi_{tt'}(k) + \varphi_{t't}(k_0) - \varphi_{tt'}(k_0) - \varphi_{t't}(k_0) + \varphi_{tt'}(k_0) + \varphi_{t't}(k') = \varphi_{tt'}(k) + \varphi_{t't}(k')\,.$$

3. We will prove that if numbers $\varphi_{tt'}(k)$ satisfy the condition in the first line of (13) then values $q_t(k) + \sum_{t' \in N(t)} \varphi_{tt'}(k)$ do not depend on k, depending only on t.

Let us consider two labellings $\bar{k}_1$ and $\bar{k}_2$, which differ one from other only at object t so that $k_1(t) = k_1$ and $k_2(t) = k_2$ and at all other objects $t' \neq t$ labels $k_1(t')$ and $k_2(t')$ are same and equal $k_0(t')$. Qualities of these two labellings are

$$G(\bar{k}_1) = \sum_{t' \in N(t)} \varphi_{tt'}(k_1) + q_t(k_1) + A,$$

$$G(\bar{k}_2) = \sum_{t' \in N(t)} \varphi_{tt'}(k_2) + q_t(k_2) + A,$$

where

$$A = \sum_{\substack{t^* t^{**} \in \Im, \\ t^* \neq t, t^{**} \neq t}} g_{t^* t^{**}}(\bar{k}_0(t^*), \bar{k}_0(t^{**})) + \sum_{\substack{t^* \in T, \\ t^* \neq t}} q_{t^*}(\bar{k}_0(t^*)) + \sum_{t^* \in N(t)} \varphi_{t^* t}(\bar{k}_0(t^*)).$$

Zero qualities $G(\bar{k}_1) = G(\bar{k}_2) = 0$ result in $\sum_{t' \in N(t)} \varphi_{tt'}(k_1) + q_t(k_1) = \sum_{t' \in N(t)} \varphi_{tt'}(k_2) + q_t(k_2)$. It means that the value $\sum_{t' \in N(t)} \varphi_{tt'}(k) + q_t(k)$ does not depend on k and depends only on t. The value $\sum_{t' \in N(t)} \varphi_{tt'}(k) + q_t(k)$ will be designated as $h(t)$.

4. We will prove that the sum $\sum_{t \in T} h(t)$ is 0. Really,

$$\sum_{t \in T} h(t) = \sum_{t \in T} \left(\sum_{t' \in N(t)} \varphi_{tt'}(k(t)) + q_t(k(t)) \right) = \sum_{tt' \notin \Im} g_{tt'}(k(t), k(t')) + \sum_{t \in T} q_t(k(t)) = 0. \quad \blacksquare$$

When a neighbourhood $\Im$ forms a connected graph on the set of objects the theorem 3 can be strengthened. Namely, it is possible to replace the condition $\sum_{t \in T} h(t) = 0$ by more strong condition $h(t) = 0$ for each $t \in T$.

Theorem 4. *Let a $(\max, +)$-problem is defined with weights $g_{tt'}(k, k')$ and $q_t(k)$, neighbourhood $\Im$ forming connected graph on the set of objects. This problem is a zero problem if and only if the potentials $\varphi_{tt'}(k)$, $tt' \in \Im$, $k \in K$, exist, which satisfy the equation system*

$$\begin{cases} g_{tt'}(k,k') = \varphi_{tt'}(k) + \varphi_{t't}(k'), & tt' \in \Im, \quad k \in K, \quad k' \in K, \\ q_t(k) = -\sum\limits_{t' \in N(t)} \varphi_{tt'}(k), & t \in T, \quad k \in K. \end{cases} \tag{22}$$

Proof. Evidently, if for the weights $g_{tt'}(k,k')$ and $q_t(k)$ there exist potentials $\varphi_{tt'}(k)$, which satisfy the conditions (22), then for the same weights $g_{tt'}(k,k')$ and $q_t(k)$ there exist also the potentials $\varphi_{tt'}(k)$ and numbers $h(t)$, which satisfy the conditions (13). They can be, particularly, the numbers $h(t)=0$.

We will show now that if for the weights $g_{tt'}(k,k')$ and $q_t(k)$ the potentials $\varphi_{tt'}(k)$ and the numbers $h(t)$ exist, which satisfy (13), then for the same weights $g_{tt'}(k,k')$ and $q_t(k)$ the potentials $\varphi_{tt'}(k)$ exist, which satisfy (22).

Let G be a graph, which forms the neighbourhood $\Im$. Since the graph is connected, there exists its spanning tree G'. Let us choose some pair (t^*,t') of objects, which are neighbours in G' and such that t^* is a leaf of the tree. Let us transform the potentials $\varphi_{t^*t'}(k)$, $\varphi_{t't^*}(k)$ and numbers $h(t')$, $h(t^*)$ in the following way:

$$\begin{aligned} \varphi_{t^*t'}(k) &:= \varphi_{t^*t'}(k) + h(t^*), \quad k \in K, \\ \varphi_{t't^*}(k) &:= \varphi_{t't^*}(k) - h(t^*), \quad k \in K, \\ h(t') &:= h(t') + h(t^*), \\ h(t^*) &:= 0. \end{aligned}$$

Each line in these expressions have to be understood as an operator, not as an equality. It means that symbols $\varphi_{t^*t'}(k)$, $\varphi_{t't^*}(k)$, $h(t')$ and $h(t^*)$ in right side of expressions designate values of corresponding variable before transformation, the same symbols in left side designating these values after transformation. The mentioned transformation of the values $\varphi_{t^*t'}(k)$, $\varphi_{t't^*}(k)$, $h(t')$ and $h(t^*)$ does not violate conditions (13). However, the value $h(t^*)$ becomes zero.

Let us exclude the object t^* from the graph G' and repeat the transformation many times until graph will contain only one object t'. At this stage the condition $\sum_{t\in T} h(t)=0$ still holds. Since for each $t\neq t'$ holds $h(t)=0$ now, condition $h(t')=0$ must be valid for t' too. ■

It follows from the theorem 3 and 4 and lemma 6 that if for two $(\max,+)$-problems with weights $g^1_{tt'}(k,k')$, $q^1_t(k)$ and $g^2_{tt'}(k,k')$, $q^2_t(k)$ the potentials $\varphi_{tt'}(k)$ exist, which satisfy the condition

$$\begin{aligned}
&g^1_{tt'}(k,k') = g^2_{tt'}(k,k') + \varphi_{tt'}(k) + \varphi_{t't}(k'), \qquad tt'\in\Im,\quad k\in K,\quad k'\in K,\\
&q^1_t(k) = q^2_t(k) - \sum_{t'\in N(t)} \varphi_{tt'}(k),\quad t\in T,\quad k\in K,
\end{aligned} \tag{23}$$

then these two problems are equivalent. Moreover and more important, if two problems with weights $g^1_{tt'}(k,k')$, $q^1_t(k)$ and $g^2_{tt'}(k,k')$, $q^2_t(k)$ are equivalent then the potentials $\varphi_{tt'}(k)$ exist, which satisfy the condition (23). The transformation (23) of the weights $g^2_{tt'}(k,k')$, $q^2_t(k)$ into the weights $g^1_{tt'}(k,k')$, $q^1_t(k)$ embraces whole set of equivalent transformation of problems. Taking into account this transformation the following general expression can be written for the power of a problem, which is equivalently transformed with potentials φ:

$$E = \sum_{tt'\in\Im} \max_{k\in K,k'\in K}\left[g_{tt'}(k,k') + \varphi_{tt'}(k) + \varphi_{t't}(k')\right] + \sum_{t\in T}\max_{k\in K}\left[q_t(k) - \sum_{t'\in N(t)}\varphi_{tt'}(k)\right]. \tag{24}$$

It is easy to see that the power is a convex function of potentials. Both expressions $g_{tt'}(k,k') + \varphi_{tt'}(k) + \varphi_{t't}(k')$ and $q_t(k) - \sum_{t'\in N(t)}\varphi_{tt'}(k)$ are linear and, consequently, convex functions of potentials $\varphi_{tt'}(k)$. The expressions $\max_{k\in K,k'\in K}\left[g_{tt'}(k,k') + \varphi_{tt'}(k) + \varphi_{t't}(k')\right]$ and $\max_{k\in K}\left[q_t(k) - \sum_{t'\in N(t)}\varphi_{tt'}(k)\right]$, in turn, are also convex functions of potentials. According to (24) the power E is a sum of convex functions and, consequently, is convex. In such way the looking for a problem with minimal power over the set of problems, which are equivalent to given one, is reduced to minimisation of convex, but non-smooth function of potentials $\varphi_{tt'}(k)$ with no restriction on potentials.

4 Algorithm of problem power minimisation

For minimisation of a problem power the method of generalised gradients (or so-called sub-gradients) was used, developed by N.Z.Shor [21]. As for the problem power minimisation the basic idea of the method consists in the following. Let us designate Φ an array of potentials $\varphi_{tt'}(k)$, $t \in T$, $t' \in N(t)$, $k \in K$. Let us define the sequence γ_i, $i = 1,2,...,$ of positive numbers so that

$$\lim_{i\to\infty} \gamma_i = 0; \quad \sum_{i=1}^{\infty} \gamma_i = \infty. \tag{25}$$

For arbitrary initial array Φ_0 the sequence Φ_i, $i = 1,2,...,$ is defined such that

$$\Phi_{i+1} = \Phi_i - \gamma_{i+1} \nabla E(\Phi_i), \tag{26}$$

where $\nabla E(\Phi_i)$ is an arbitrary sub-gradient of the power E in the point Φ_i. The main property of sub-gradient descent consists in that the limit of the sequence $E(\Phi_0)$, $E(\Phi_1)$, $\ldots$ equals $\min_{\Phi} E(\Phi)$ and this limit does not depend on initial array Φ_0.

For the power minimisation an array Φ can be treated as a vector in linear space $\Re^{2|\Im\times K|}$, potentials $\varphi_{tt'}(k)$ being its components. It is known that the sub-gradient of convex function is not obligatory unambiguously defined. One of the sub-gradients of the power E in the point Φ_i can be obtained in the following way:

1. For each object $t \in T$ choose a label $k^*(t)$, for which

$$q_t(k^*(t)) - \sum_{t' \in N(t)} \varphi^i{}_{tt'}(k^*(t)) = \max_{k\in K}\left[q_t(k) - \sum_{t'\in N(t)} \varphi^i{}_{tt'}(k) \right].$$

If this condition is fulfilled for several different labels choose any of them.

2. For each pair $tt' \in \Im$ of neighbours choose a pair $k^*{}_{tt'}(t)$, $k^*{}_{tt'}(t')$ of labels, for which

$$g_{tt'}(k^*{}_{tt'}(t), k^*{}_{tt'}(t')) + \varphi^i{}_{tt'}(k^*{}_{tt'}(t)) + \varphi^i{}_{t't}(k^*{}_{tt'}(t')) = \max_{k\in K,\, k'\in K} \left(g_{tt'}(k,k') + \varphi^i{}_{tt'}(k) + \varphi^i{}_{t't}(k') \right).$$

Similarly as in previous step if this condition fulfils for several label pairs choose any of them.

3. Define the components of the sub-gradient in the following way:

$$\nabla\varphi_{tt'}(k)=\begin{cases} 1, & if \quad k^*_{tt'}(t)=k, \quad k^*(t)\neq k, \\ -1, & if \quad k^*_{tt'}(t)\neq k, \quad k^*(t)=k, \\ 0 & otherwise. \end{cases}$$

5. Intermediate discussion.

Now it becomes possible to summarise above described results and compare them with what has been known up to now.

A problem of $(\max,+)$-labelling consists in the following: for given sets T, $\Im\subset T\times T$, K and given functions $g_{tt'}:K\times K\to R$, $tt'\in\Im$ and $q_t:K\to R$, $t\in T$, the labelling $\bar{k}^*:T\to K$ with maximal quality

$$G(\bar{k})=\sum_{tt'\in\Im}g_{tt'}\left(k(t),k(t')\right)+\sum_{t\in T}q_t\left(k(t)\right)$$

has to be found.

The main idea of proposed approach is closely connected with other labelling problem that can be called $(\vee,\&)$-problem. It consists in that for given sets T, $\Im\subset T\times T$, K and given functions $g_{tt'}:K\times K\to\{0,1\}$, $tt'\in\Im$, and $q_t:K\to\{0,1\}$, $t\in T$, the question has to be answered, whether such labelling $\bar{k}^*:T\to K$ exists that

$$G(\bar{k}^*)=\mathop{\&}_{tt'\in\Im}g_{tt'}\left(k^*(t),k^*(t')\right)\,\&\,\mathop{\&}_{t\in T}q_t\left(k^*(t)\right)=1.$$

If such labelling exists the $(\vee,\&)$-problem is said to have a positive solution.

An input data of an $(\vee,\&)$-problem or, simply, $(\vee,\&)$-problem will be called locally consistent if for each triple $tt'\in\Im$, $k\in K$ an equality

$$q_t(k)=\bigvee_{k'}g_{tt'}(k,k').$$

holds. An $(\vee,\&)$-problem will be called empty if $q_t(k)=0$ for each $t\in T$, $k\in K$. We will say also that $(\vee,\&)$-problem with weights $g_{tt'}(k,k')$ and $q_t(k)$ includes $(\vee,\&)$-problem with weights $g'_{tt'}(k,k')$ and $q'_t(k)$ if $g'_{tt'}(k,k')\le g_{tt'}(k,k')$ and $q'_t(k)\le q_t(k)$ for each $t\in T$, $t'\in N(t)$, $k\in K$, $k'\in K$. It is evident that each positively solvable $(\vee,\&)$-problem includes non-empty consistent part. Certainly, a converse proposition is not valid.

Let for a given $(\max,+)$-problem with weights $g_{tt'}(k,k')\in R$, $q_t(k)\in R$ the corresponding $(\vee,\&)$-problem with numbers $\overline{g}_{tt'}(k,k')\in\{0,1\}$, $\overline{q}_t(k)\in\{0,1\}$ is defined in such way that

$$\overline{g}_{tt'}(k,k')=1,\ \text{if}\ g_{tt'}(k,k')=\max_{l,l'} g_{tt'}(l,l'),$$

$$\overline{g}_{tt'}(k,k')=0,\ \text{if}\ g_{tt'}(k,k')<\max_{l,l'} g_{tt'}(l,l'),$$

$$\overline{q}_t(k)=1,\ \text{if}\ q_t(k)=\max_{l} q_t(l),$$

$$\overline{q}_t(k)=0,\ \text{if}\ q_t(k)<\max_{l} q_t(l).$$

An idea, which unifies the proposed and known approaches, consists in such equivalent transforming of a given $(\max,+)$-problem that corresponding $(\vee,\&)$-problem becomes positively solvable. It is what was denoted in the present paper as an equivalent transformation of the problem into trivial one as well as in some previous papers of one of the authors. Certainly, not all $(\max,+)$-problems are equivalent to trivial one and so a domain of application of the idea is restricted with the problems, which do have a trivial equivalent.

The proposed approach differs from known ones because it trivialise surely each $(\max,+)$-problem, for which such trivial equivalent exists at all, whereas the known approaches do not provide such certainty. The known algorithms transform an initial $(\max,+)$-problem into such form that corresponding $(\vee,\&)$-problem contains non-empty locally consistent part. As for proposed algorithm it transforms an initial problem into a problem with minimal power and so is stronger than known algorithms. Namely, if a $(\max,+)$-problem is such that corresponding $(\vee,\&)$-problem contains a non-empty locally consistent part it not obligatory minimises a power in its equivalence class. However, the $(\vee,\&)$-problem, which corresponds to $(\max,+)$-problem with minimal power, surely contains non-empty locally consistent part. Later on when experiments with algorithms will be described a plain example will be shown that makes clear the last two statement.

Besides of this main result it is appropriate to mention additional by-result concerning a solution of $(\vee,\&)$-problems on the base of equivalent transformation of $(\max,+)$-problems.

Due to *NP*-completeness of class of all possible $(\vee,\&)$-problems it is hardly possible necessary and sufficient and easily recognisable condition of positive solvability of the problem.

However, there is well-known necessary, not sufficient, above mentioned condition, which serves as a base for several applied algorithms [23]. Namely, for positive solvability of $(\vee,\&)$-problem it is necessary for it to contain a non-empty locally consistent part. Idea of power minimisation shows that this condition can be strengthened in the following way.

Let $g_{tt'}(k,k')$, $tt'\in\Im$, $k\in K$, $k'\in K$, be numbers 0 or 1, which define $(\vee,\&)$-problem to calculate the value

$$G(\vee,\&)=\bigvee_{\bar{k}\in K^T}\mathop{\&}_{tt'\in\Im} g_{tt'}\big(k(t),k(t')\big). \qquad (27)$$

Let the same numbers define $(\max,+)$-problem of calculating the value

$$G(\max,+)=\max_{\bar{k}\in K^T}\sum_{tt'\in\Im} g_{tt'}\big(k(t),k(t')\big). \qquad (28)$$

It is evident that for positive solvability of an $(\vee,\&)$-problem (27) it is necessary that a $(\max,+)$-problem (28) minimises power in its equivalence class. It is evident also that this necessary condition is not weaker than above defined one. In fact, it is stronger and later on it will be shown an example of non-empty consistent $(\vee,\&)$-problem, which has no positive solution, which absence can be easily deduced using an equivalent transformation of corresponding $(\max,+)$-problem.

Though development of the algorithm needed somewhat extensive analysis of formal properties of $(\max,+)$-problems, the final result has appeal due to its unexpected simplicity, possibility of paralleling and, may be, due to other properties. At the same time, as any algorithm that intends to cope with vast problem class it can be less preferable for certain special problem subclasses as compared with algorithms fitted just for the case. In this sense the above mentioned algorithm is only an illustration of power minimisation approach that has to serve as a starting point at developing the algorithms for specific application. We will show how the approach has to be applied when a set of objects is not an abstract set with arbitrary neighbourhood, but a field of vision with a specific and clearly visible two-dimensional structure.