TEXTURE SEGMENTATION OF IMAGES ON THE BASIS OF MARKOV RANDOM FIELDS

IVAN KOVTUN

International Research and Training Center of Information Technologies and Systems, Kiev, Ukraine, kovtun@image.kiev.ua
Dresden University of Technology, kovtun@ics.inf.tu-dresden.de

Abstract. This paper presents a new model of textured images consisting of several textures. The segmentation and each texture are described by their own Markov random fields. Such a model allows to avoid any additional restrictions on the random fields, such as autoregressivness or gaussianity. Different segmentation problem formulations based on this model and their solutions are considered as well.

1. Introduction

A texture is an important characteristic of images of natural objects and an important element of visual perception. A considerable quantity of investigations [1] is devoted to the computer analysis of texture images. In the sphere of these investigations the problem of image segmentation by texture features takes an essential part. One may separate the following three main groups in the long-term investigations of the problem.

1. We include into the first group of works the vast researche that were done in this area before the publication of the fundamental work by Gemans [2]. The main difference between these investigations consists in that various texture features are postulated on the basis of some intuitive heuristic considerations. These features are calculated for each pixel on the basis of some neighborhood. Thus these algorithms are also called window-based algorithms. Algorithms of texture segmentation are constructed on the base of informal considerations, not as the solution of a formally set problem. It is supposed that the texture feature varies slightly within one segment and takes essentially different values on different segments. Rather often the algorithms of this class take the form of "expansion" or "merging" of areas in the image. The following algorithms are also rather widespread: the boundaries between two segments are selected on the basis of chosen features, then the areas are constructed in one or another way in compliance with the selected boundaries. Such heuristic works were done both before the Gemans' work [2] and after it. Having classified these works to before-Gemans period, we want to stress the fundamental importance of Gemans' works in which the texture segmentation problem got the precise mathematical formulation.

2. In Gemans' work the texture segmentation problem is formulated on the basis of Markov models of images and consists in the search for realization of the hidden Markov field with the maximum a posterior probability. This problem comes to

ISSN 1430-211X, TUD-FI03 Mai 2003.

extremely complex optimization problems except several special cases when, for example, the Markov field is autoregressive or gaussian. For solving the general problem the method of simulated annealing is proposed. This method was successfully applied before for practical solving other theoretically hopeless problems [3, 4]. Probably just after Gemans' work the method of simulated annealing became wide applicable in the processing and recognition of images. However, it is known that the simulated annealing does not determine unambiguously the algorithm of finding the global optimum, it only indicates that the sought algorithm belongs to a vast class of algorithms. The algorithms of this class differ from one another only in the process of changing a special parameter, which is known as the "temperature" of the annealing. Only under the certain dependence of this temperature on the time of annealing the algorithms of the simulated annealing ensures the convergence to the global maximum of the function being optimized. The choice of such dependence was the main problem while using the recommendations of Gemans for solving real-world problems.

3. Later the structural recognition was investigated from the point of view of the Bayes theory of statistical solutions, and these new results in the sphere of structural recognition determined further progress in the texture segmentation field [5, 6, 7]. It became known that searching the segmentation with the maximum a posterior probability is the solution to only one of several possible Bayes recognition problems. Moreover, such solution is simply unwise in a number of applications. It follows from the form of the penalty function, which is absolutely unnatural in this case. These results gave opportunity to construct new algorithms of recognition of hidden Markov fields [8, 9, 10, 11, 12]. These algorithms are very close by their nature to the algorithms of stochastic relaxation of Gemans, but at that it appeared unnecessary to solve a complex and at present time unsolved problem about the speed of decreasing the "temperature". The problems of hidden Markov fields recognition, specifically, the problems of image texture segmentation appeared to be a particular case of the more general problem of modelling and generating Markov random fields. In this sphere of investigations the problem of image segmentation has lost its specific character.

The present paper presents an attempt to isolate the texture segmentation problem from the general class of problems of generating and modelling of Markov random fields. We shall construct the algorithms of texture segmentation that take into consideration the specific character of this subclass of problems in the general problem of generation of Markov random fields. These algorithms will be constructed not on the basis of Markov model for images, but on the basis of a model for such images that are constructed from several segments, and each of these segments is a random realization of Markov random field, that is unique for each segment.

In the second and third sections the main definitions that are necessary for precise formulation of the problem are given. In the fourth section the model of the image consisting of several textures is introduced. In the fifth section possible texture segmentation problems within Bayes recognition theory are formulated. In the sixth section the method of reduction of these problems to the problems of generation of Markov random fields is offered. In the seventh section the algorithm of Markov random field generation for the formulated model of texture image is described. In the eighth section we present the examples of practical realization of the proposed algorithm.

2. The basic definitions

Basic definitions are introduced in this section. Later we operate with such notions as ***a field of vision, a pixel, a palette, a color, an image, a label and a structure of the field of vision***. Most of these notions are intuitively comprehensible, but we introduce them for clarity.

Let ***a field of vision*** T be an arbitrary finite set. Elements of the field of vision are called ***pixels***. One of the most frequently encountered example of the field of vision is a rectangular area of the two-dimensional integer lattice $\{(i,j)\,|0 \le i < I, 0 \le j < J\}$. Without loss of generality we may assume that the elements of the field of vision are natural numbers
$T = \{1, 2, ..., m\}$.

Let ***a palette*** X be an arbitrary finite set, as well as a field of vision. The elements of the palette X are called ***colors***. Without loss of generality we may assume that $X = \{1, 2, ..., n\}$.

A function $x_T : T \to X$ is called ***an image***. The restriction of this function on a subset of the field of vision $\tau \subset T$ is denoted by x_τ, and the value of the function x_T in the pixel $t \in T$ is denoted by x_t.

Let ***a labelling*** of the field of vision T by l segments be a function $k_T : T \to \{1, 2, ..., l\}$. At that the set $\{1, 2, ..., l\}$ is called ***a set of labels*** and denoted by the symbol K.

The last two definitions are almost identical. Introduction of two definitions instead of one is due to the fact that the notions "image" and "labelling" carry different semantic capacity, therefore their separation improves the comprehension of the stated material. The labelling and the image have the same properties as a function that maps the field of vision into a finite set.

Let ***a structure of the field of vision*** T be a set of subsets $\Im \subseteq \{\tau|\tau \subseteq T\}$.

Note that the set $\Im$ does not necessary contain all subsets of the field of vision. Usually the elements of the structure of the field of vision consist of just pairs of pixels.

3. Random field, Markov random field

A random value $\aleph$ defined on the set of all images $X^T = \{x_T|x_T : T \to X\}$ is called ***a random field***. It is impossible to specify directly a probability distribution on the set of all images due to their huge number. Because of this, the random field $\aleph$ is usually considered as a family of random values $\aleph = \{\aleph_t|t \in T\} = \{\aleph_1, \aleph_2, ..., \aleph_m\}$. Each random value $\aleph_t$ corresponds to one pixel t and takes values from the set X. At that the random values $\aleph_1, \aleph_2, ..., \aleph_m$ are not necessarily independent. For the given image $x_T : T \to X$ the probability of the event $(x_1 = \aleph_1, x_2 = \aleph_2, ..., x_m = \aleph_m)$ will be denoted by $p\,(x_T)$. For the subset of pixels $\tau \subseteq T$ we denote the probability of the event $\{x_t = \aleph_t|t \in \tau\}$ by $p\,(x_\tau)$.

In the present article all random fields are assumed to be Markov, which means that their probability distributions are given by:

$$p\,(x_T) = z \cdot \exp \sum_{\tau \in \Im} \varphi_\tau\,(x_\tau), \tag{1}$$

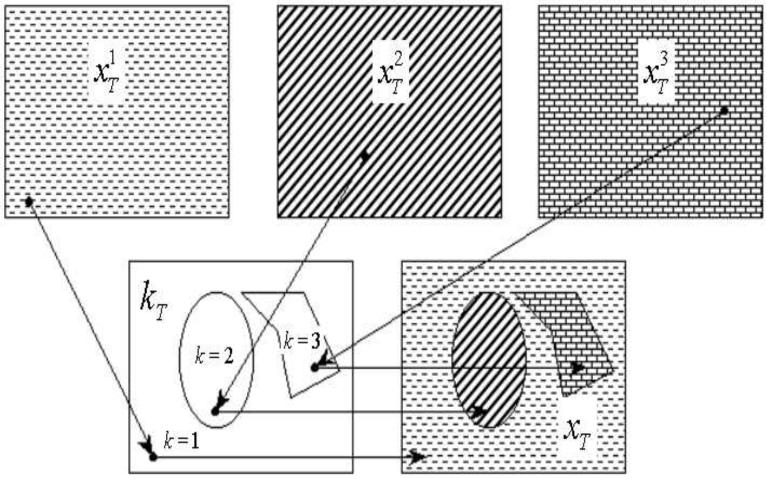


Figure 1. General model of texture image constructed in accordance with the formula (3).

where $z^{-1} = \sum\limits_{x_T \in X^T} \exp \sum\limits_{\tau \in \Im} \varphi_\tau (x_\tau)$ is a normalizing factor, $\Im$ is a structure of the field of vision, φ_τ are functions depending on the variables $x_\tau = \{x_t | t \in \tau\}$. The functions φ_τ are called Gibbs potentials.

It was shown in [8, 9, 10, 11] that Markov random fields could be used for specification of textures. Besides, the algorithm for generation of Markov random fields according to the probability distribution (1) is known. This algorithm is based on the fact that the probability distribution of the labels in one pixel is easily computed under condition that labels in all other pixels are fixed:

$$p\left(x_t | x_{T\setminus t}\right) = c \cdot \exp \sum_{\substack{\tau \in \Im \\ t \in \tau}} \varphi_\tau (x_\tau), \tag{2}$$

where $c^{-1} = \sum\limits_{x_t \in X} \exp \sum\limits_{\substack{\tau \in \Im \\ t \in \tau}} \varphi_\tau \left(x_{\tau \setminus t}, x_t\right)$ is a normalizing factor providing the equality $\sum\limits_{x_t} p\left(x_t | x_{T\setminus t}\right) = 1$.

The algorithm of generating consists in the following. At the first step of the algorithm an arbitrary image x_T is chosen. Further the following operation iterates: an arbitrary pixel of the field of vision $t \in T$ is chosen and the color in this pixel is generated according to the probability distribution (2).

4. Introduction of the basic model of the textured image

Let l random fields be given on the set of all images X^T. The probability distribution of these random fields will be denoted by $p^1(x_T), p^2(x_T), ..., p^l(x_T)$ correspondingly. Each of these random fields corresponds to a certain type of texture. The upper index denote the number of the texture. Let also a random field on the set of all labellings of the field of vision T by l segments be given. We denote the probability of the labelling $k_T : T \to K$ by $p^{lab}(k_T)$.

Let ***a textured image*** x_T be an image obtained as a result of the following random process. Images $x_T^1, x_T^2, ..., x_T^l$ and the labelling k_T are generated independently according to the probability distributions $p^1(x_T), p^2(x_T), ..., p^l(x_T)$ and $p^{lab}(k_T)$ correspondingly. Then the textured image x_T is constructed from the obtained images following the rule: if the pixel t of the labelling is labelled by $k_t \in K = \{1, 2, ..., l\}$, then the color from the k -th texture image x_T^k is carried to the corresponding pixel x_t of the resulting image (see figure 1). More exactly the color x_t in each pixel $t \in T$ is computed by the formula

$$\forall t \in T : x_t = x_t^{k_t}. \tag{3}$$

From the construction we obtain the formula for simultaneous distribution of the probabilities of images x_T^1, x_T^2, ..., x_T^l, k_T and x_T :

$$p\left(x_T^1, \dots, x_T^l, k_T, x_T\right) = \begin{cases} 0 & \exists t \in T : x_t \neq x_t^{k_t} \\ p^1\left(x_T^1\right) \cdot \dots \cdot p^l\left(x_T^l\right) \cdot p^{lab}(k_T) & \forall t \in T : x_t = x_t^{k_t} \end{cases}. \tag{4}$$

On the basis of the described model it is necessary to restore the labelling k_T provided that the image x_T is known.

In this work we assume that random fields are Markov random fields. The model of textured image will be described by $l+1$ structures of the field of vision $\Im^1, \Im^2, ..., \Im^l, \Im^{lab}$ and for every $i \in \{1, 2, ..., l, lab\}$, for every $\tau \in \Im^i$ the functions φ_τ^i specify Gibbs potentials. The probability distribution of the image of i-th texture x_T^i is given by the formula $p^i\left(x_T^i\right) = z^i \cdot \exp \sum\limits_{\tau \in \Im^i} \varphi_\tau^i\left(x_\tau^i\right)$, and the probability distribution of the label k_T is equal to $p^{lab}(k_T) = z^{lab} \cdot \exp \sum\limits_{\tau \in \Im^{lab}} \varphi_\tau^{lab}(k_\tau)$.

In the works [8, 9, 10, 11, 12] G.L.Gimel'farb and A.Zalesny consider texture segmentation of images as a particular case of generating Markov random fields. At that the color of each pixel consists of two components (x, k) that are the signal value and the label. At that it is implicitly suggested that the structure of the field of vision is the same for different textures. The advantage of the just introduced model is that the labelling and each texture are characterized by their own Markov random fields that are independent. This allows to modify easily the recognition system. One can introduce a new texture or eliminate existing one easily within the framework of the introduced model. Besides, in the given model it is easy to allow for a priori knowledge about the labelling.

5. Formulation of the segmentation problem as the Bayesian problem of recognition

It is necessary to find the labelling k_T when the simultaneous probability distribution $p\left(x_T^1, x_T^2, \dots, x_T^l, k_T, x_T\right)$ and the image x_T are known. Let us consider various problem formulations for searching for the labelling k_T.

1) It is necessary to find the most probable values of all hidden parameters of the model, namely, the images $x_T^1, x_T^2, \dots, x_T^l$ and the labelling k_T, according to their a posterior probability distribution $p\left(x_T^1, x_T^2, \dots, x_T^l, k_T | x_T\right)$. Thus, it is necessary to find

$$\left(x_T^{*1}, x_T^{*2}, \dots, x_T^{*l}, k_T^*\right) = \operatorname*{arg\,max}_{x_T^1, x_T^2, \dots, x_T^l, k_T} p\left(x_T^1, x_T^2, \dots, x_T^l, k_T | x_T\right) =$$

$$= \arg\max_{x_T^1, x_T^2, \ldots, x_T^l, k_T} \frac{p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T, x_T\right)}{p\left(x_T\right)} =$$

$$= \arg\max_{x_T^1, x_T^2, \ldots, x_T^l, k_T} p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T, x_T\right) =$$

$$= \arg\max_{x_T^1, x_T^2, \ldots, x_T^l, k_T} \begin{cases} 0 & \exists t \in T : x_t \neq x_t^{k_t} \\ z^{lab} \cdot \exp \sum\limits_{\tau \in \Im^{lab}} \varphi_\tau^{lab}(k_\tau) \cdot \prod\limits_{i=1}^{l} z^i \cdot \exp \sum\limits_{\tau \in \Im^i} \varphi_\tau^i(x_\tau^i) & \forall t \in T : x_t = x_t^{k_t} \end{cases} =$$

$$= \arg\max_{x_T^1, x_T^2, \ldots, x_T^l, k_T} \begin{cases} 0 & \exists t \in T : x_t \neq x_t^{k_t} \\ \sum\limits_{\tau \in \Im^{lab}} \varphi_\tau^{lab}\left(k_\tau\right) + \sum\limits_{i=1}^{l} \sum\limits_{\tau \in \Im^i} \varphi_\tau^i\left(x_\tau^i\right) & \forall t \in T : x_t = x_t^{k_t} \end{cases} =$$

$$\arg\max_{\substack{x_T^1, x_T^2, \ldots, x_T^l, k_T \\ \forall t \in T : x_t = x_t^{k_t}}} \Big(\sum_{\tau \in \Im^{lab}} \varphi_\tau^{lab}(k_\tau) + \sum_{i=1}^{l} \sum_{\tau \in \Im^i} \varphi_\tau^i(x_\tau^i) \Big).$$

After that one may take the obtained labelling k_T^* as a result of recognition. In the case when the order of the structure of the field of vision is more than one, the solution to the problem is unknown.

However, the solution of the problem has no statistical sense since the most probable values of all hidden parameters of the system are used for finding the labelling (see [5]). The following statement of the problem is more reasonable.

2) Given an image x_T, it is necessary to find the most probable labelling in accordance with its a posterior probability distribution $p\left(k_T | x_T\right)$. This is the most widely used formulation of the problem, known as the maximum a posteriori estimation.

$$k_T^* = \arg\max_{k_T} p\left(k_T | x_T\right) = \arg\max_{k_T} \sum_{x_T^1, x_T^2, \ldots, x_T^l} p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right) =$$

$$= \arg\max_{k_T} \sum_{x_T^1, x_T^2, \ldots, x_T^l} \frac{p\left(x_T^1, x_T^2, .., x_T^l, k_T, x_T\right)}{p\left(x_T\right)} =$$

$$= \arg\max_{k_T} \frac{\sum\limits_{x_T^1, x_T^2, \ldots, x_T^l} p\left(x_T^1, x_T^2, .., x_T^l, k_T, x_T\right)}{p\left(x_T\right)} =$$

$$= \arg\max_{k_T} \sum_{x_T^1, x_T^2, \ldots, x_T^l} p\left(x_T^1, x_T^2, .., x_T^l, k_T, x_T\right) =$$

$$= \arg\max_{k_T} \sum_{\substack{x_T^1, x_T^2, \ldots, x_T^l \\ \forall t \in T : x_t = x_t^{k_t}}} \Big(z^{lab} \cdot \exp \sum_{\tau \in \Im^{lab}} \varphi_\tau^{lab}(k_\tau) \cdot \prod_{i=1}^{l} z^i \cdot \exp \sum_{\tau \in \Im^i} \varphi_\tau^i(x_\tau^i) \Big) =$$

$$= \arg\max_{k_T} \sum_{\substack{x_T^1, x_T^2, \ldots, x_T^l \\ \forall t \in T : x_t = x_t^{k_t}}} \exp \Big(\sum_{\tau \in \Im^{lab}} \varphi_\tau^{lab}(k_\tau) + \sum_{i=1}^{l} \sum_{\tau \in \Im^i} \varphi_\tau^i(x_\tau^i) \Big).$$

This problem is more complicated then previous one as it is impossible even to compare posterior probabilities of two arbitrary labellings.

3) On the other hand it is also unnatural to find the most probable labelling, i.e. to solve the problem 2) (though it is a common practice). It simply means to solve

the Bayesian recognition problem with the simplest penalty function $W(k_T, k_T^*) = \begin{cases} 0 & k_T = k_T^* \\ 1 & k_T \neq k_T^* \end{cases}$. According to this penalty function the segmentation that differs from the real one only in one pixel is as bad as the segmentation that differs from the real one in all pixels. It would be more reasonable to consider the Bayesian problem with the additive penalty function $W(k_T, k_T^*) = \sum_{t \in T} w(k_t, k_t^*)$. To solve this problem it is necessary to find the posterior probability distribution of the labelling in each pixel [5]:

$$p_t(k) = \sum_{\substack{k_T \\ k_t = k}} p(k_T | x_T).$$

If $w(k, k^*) = \begin{cases} 0 & k = k^* \\ 1 & k \neq k^* \end{cases}$, then the penalty is proportional to the number of misrecognized pixels. The solution of the Bayesian problem takes the following form:

$$k_t = \arg\max_k p_t(k).$$

The computational complexity of the problem makes it practically unsolvable.

6. The reduction of above-stated problems to the problems of generation

Let us replace the problems 1), 2) and 3) by the problems 1*), 2*) and 3*) correspondingly. We replace the condition of finding the most probable value by the generation of the value with a given probability distribution. We obtain the following three problems:

1*) It is necessary to construct an algorithm for generating the images $x_T^1, x_T^2, \ldots, x_T^l$ and the labelling k_T in accordance with their a posterior probability distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right)$.

2*) It is necessary to construct an algorithm for generating the labelling k_T in accordance with its a posterior probability distribution $p(k_T | x_T)$.

3*) It is necessary to construct an algorithm for generating the label k in the pixel t in accordance with its a posterior probability distribution $p_t(k)$.

Knowing the answer to any of the problems 1), 2) or 3) does not allow us to solve the other two problems. As for the problems 1*), 2*) and 3*) it is all the opposite. Being able to generate the images $x_T^1, x_T^2, \ldots, x_T^l$ and the label k_T according to the distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right)$, i.e. knowing the answer to the problem 1*), it is also easy to construct the solutions of the problems 2*) and 3*). To generate the label k_T according to the probability distribution $p(k_T | x_T) = \sum_{x_T^1, x_T^2, \ldots, x_T^l} p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right)$ it is sufficient to generate the images $x_T^1, x_T^2, \ldots, x_T^l$ and the label k_T according to the distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right)$, then take the label k_T as an output of the algorithm 2*). Later on, to generate the label in the pixel t according to the distribution $p_t(k)$ one may generate the labelling k_T according to the distribution $p(k_T | x_T)$ and take the label k_t in the pixel t as an output of the algorithm 3*).

Note that having generated the label in the pixel t with the distribution $p_t(k)$ several times it is possible to estimate this probability distribution and to use the obtained estimation for solving the problem 3). Thus, the solution of the problem 3*) allows to solve the problem 3) with a certain degree of reliability. In the same way, though less obviously, the solution of the problem 2*) allows to solve the problem 2), and the solution of the problem 1*) allows to solve the problem 1). In order to solve the problem 2) it is necessary to generate the labelling in accordance with the probability distribution $p(k_T|x_T)$ many times, then take the labelling that has met most frequently as the solution to the problem 2). Similarly, in order to solve the problem 1) it is necessary to generate the images $x_T^1, x_T^2, \ldots, x_T^l$ and the labelling k_T in accordance with the probability distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T|x_T\right)$ many times and take the set of images $x_T^1, x_T^2, \ldots, x_T^l$ and k_T that have met most frequently as the solution to the problem 1). At that the ability to generate is used for estimation of the probability distribution. It is easy to see that it is impossible to solve the problems 1) and 2) using the described above algorithm since the number of required samples becomes huge. Perhaps it occurs just because the problems 1) and 2) are posed as unwise Bayesian recognition problems.

Further we solve only the problem 3), having constructed the solution of the problem 3*).

7. Description of the generation method and its improvement

It is shown in the present section that the probability distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T|x_T\right)$ in the described model of textured image is given by a Markov random field. Therefore, to generate the images $x_T^1, x_T^2, \ldots, x_T^l$ and the labelling k_T according to the distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T|x_T\right)$ it is possible to use the standard algorithm for generating Markov random fields that was described in Section 3. It is shown further that computational complexity of generation of the label in one pixel is proportional to $|X|^{l-1} \cdot l$. It means that the introduction of each additional texture slows down the algorithm of generation $|X|$ times. Though it is possible to modify the algorithm of generation in such a way that computational complexity of generation of the label in one pixel will depend linearly on the number of textures. Namely, it will be proportional to $|X| \cdot l$. The algorithm of such acceleration is described in the present section as well.

Let us show first that the probability distribution $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T|x_T\right)$ is markovian on the field of vision T.

Indeed, a posterior probability of the images $x_T^1, x_T^2, \ldots, x_T^l$ and the labelling k_T is equal to

$$p\left(x_T^1,\ldots,x_T^l,k_T|x_T\right)=\begin{cases} 0 & \exists t\in T: x_t\neq x_t^{k_t} \\ z\cdot\exp\Big(\sum\limits_{\tau\in\Im^{lab}}\varphi_\tau^{lab}(k_\tau)+\sum\limits_{i=1}^{l}\sum\limits_{\tau\in\Im^i}\varphi_\tau^i(x_\tau^i)\Big) & \forall t\in T: x_t=x_t^{k_t} \end{cases}, \tag{5}$$

where z is a normalizing factor.

Now let us specify the palette in each pixel in such a way that in the pixel $t \in T$ the palette consists of colors $x^1, x^2, \ldots, x^l$ and the label k, that go with the label x_t in the observed image x_T. More precisely, the palette in the pixel t equals $X_t = \left\{\left(x^1, x^2, \ldots, x^l, k\right) \in X^l \times K | x^k = x_t\right\}$. Thus the set of all possible images in the Markov field coincides with the set of images which satisfy the condition

$$\forall t \in T : x_t = x_t^{k_t}. \tag{6}$$

Allowing for the condition (6) the formula (5) can be rewritten in the form

$$p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right) = z \cdot \exp\Big(\sum_{\tau \in \Im^{lab}} \varphi_\tau^{lab}(k_\tau) + \sum_{i=1}^{l} \sum_{\tau \in \Im^i} \varphi_\tau^i(x_\tau^i)\Big). \tag{7}$$

Therefore, we deal with Markov random field.

It follows from formula (7) that the structure of the field of vision is equal to $\Im = \Im^1 \cup \Im^2 \cup \cdots \cup \Im^l \cup \Im^{lab}$, and Gibbs potentials are defined by the formula

$$\forall \tau \in \Im : \varphi_\tau\left(x_\tau^1, x_\tau^2, \ldots, x_\tau^l, k_\tau\right) = \begin{cases} \sum\limits_{i:\tau \in \Im^i} \varphi_\tau^i\left(x_\tau^i\right) + \varphi_\tau^{lab}\left(k_\tau\right) & \tau \in \Im^{lab} \\ \sum\limits_{i:\tau \in \Im^i} \varphi_\tau^i\left(x_\tau^i\right) & \tau \notin \Im^{lab} \end{cases}.$$

Thus, $p\left(x_T^1, x_T^2, \ldots, x_T^l, k_T | x_T\right)$ is really a Markov probability distribution. To generate the random field according to the probability distribution it is necessary to generate the vector $\left(x_t^1, x_t^2, \ldots, x_t^l, k_t\right) \in X_t$ in the pixel t according to the following conditional probability distribution

$$p\left(x_t^1, x_t^2, \ldots, x_t^l, k_t | x_{T\setminus t}^1, x_{T\setminus t}^2, \ldots, x_{T\setminus t}^l, k_{T\setminus t}, x_T\right) =$$

$$= c \cdot \exp \sum_{\substack{\tau \in \Im \\ t \in \tau}} \varphi_\tau\left(x_\tau^1, x_\tau^2, \ldots, x_\tau^l, k_\tau\right) =$$

$$= c \cdot \exp\Big(\sum_{\substack{\tau \in \Im^{lab} \\ t \in \tau}} \varphi_\tau^{lab}(k_\tau) + \sum_{i=1}^{l} \sum_{\substack{\tau \in \Im^i \\ t \in \tau}} \varphi_\tau^i(x_\tau^i)\Big) =$$

$$= c' \cdot \prod_{i=1}^{l} p(x_t^i | x_{T\setminus t}^i) \cdot p(k_t | k_{T\setminus t}), \tag{8}$$

where c and c' are normalizing factors.

Direct use of the formula (8) for generating the label in one pixel requires $|X_t| = |X|^{l-1} \cdot |K| = |X|^{l-1} \cdot l$ operations since for each vector $\left(x_t^1, x_t^2, \ldots, x_t^l, k_t\right) \in X_t$ it is necessary to calculate the probability $p\left(x_t^1, x_t^2, \ldots, x_t^l, k_t | x_{T\setminus t}^1, x_{T\setminus t}^2, \ldots, x_{T\setminus t}^l, k_{T\setminus t}, x_T\right)$. However, it is possible to generate the signal in the pixel with probability distribution (8), with computational complexity being proportional to $|X| \cdot l$.

The improved generation is based on the usage of the following simple formula of the probability theory: $p\,(a, b) = p\,(a) \cdot p\,(b|a)$. Due to this formula it is possible to generate a random pair (a, b) in accordance with the probability distribution $p\,(a, b)$ in such a way: at first a is generated in accordance with the distribution $p\,(a)$, then b is generated in accordance with the probability $p\,(b|a)$ given a.

This procedure results in the following algorithm of generation. We generate the label k_t in accordance with the distribution $p\left(k_t | x_{T\setminus t}^1, \ldots, x_{T\setminus t}^l, k_{T\setminus t}, x_T\right)$, then the colors $x_t^1, x_t^2, \ldots, x_t^l$ are generated in accordance with the distribution $p\left(x_t^1, \ldots, x_t^l | x_{T\setminus t}^1, \ldots, x_{T\setminus t}^l, k_T, x_T\right)$.

Let us compute the probability of the label in the given pixel

$$p\left(k_t | x_{T\setminus t}^1, x_{T\setminus t}^2, \ldots, x_{T\setminus t}^l, k_{T\setminus t}, x_T\right) =$$

$$= \sum_{\substack{x_t^1, x_t^2, \ldots, x_t^l \\ x_t^{k_t} = x_t}} p\left(x_t^1, x_t^2, \ldots, x_t^l, k_t | x_{T\setminus t}^1, x_{T\setminus t}^2, \ldots, x_{T\setminus t}^l, k_{T\setminus t}, x_T\right) =$$

$$= c' \cdot \sum_{\substack{x_t^1, x_t^2, \ldots, x_t^l \\ x_t^{k_t} = x_t}} \prod_{i=1}^{l} p\left(x_t^i | x_{T\setminus t}^i\right) \cdot p\left(k_t | k_{T\setminus t}\right) =$$

$$= c' \cdot p\left(k_t | k_{T\setminus t}\right) \cdot p\left(x_t^{k_t} = x_t | x_{T\setminus t}^{k_t}\right) \cdot \prod_{i \neq k_t} \sum_{x_t^i} p\left(x_t^i | x_{T\setminus t}^i\right) =$$

$$= c' \cdot p\left(k_t | k_{T\setminus t}\right) \cdot p\left(x_t^{k_t} = x_t | x_{T\setminus t}^{k_t}\right). \tag{9}$$

Computational complexity of the algorithm of generation of the label in the given pixel following the formula (9) is proportional to $|X| \cdot l$. For each k_t computation of the probabilities $p\left(x_t^{k_t} = x_t | x_{T\setminus t}^{k_t}\right)$ requires $|X|$ operations. The probability distribution of the colors $x_t^1, x_t^2, \ldots, x_t^l$ under fixed label k_t is given by

$$p\left(x_t^1, x_t^2, \ldots, x_t^l | x_{T\setminus t}^1, x_{T\setminus t}^2, \ldots, x_{T\setminus t}^l, k_T, x_T\right) =$$

$$= c'' \cdot \prod_{i=1}^{k_t - 1} p\left(x_t^i | x_{T\setminus t}^i\right) \cdot p\left(x_t^{k_t} = x_t | x_{T\setminus t}^{k_t}\right) \cdot \prod_{i=k_t+1}^{l} p\left(x_t^i | x_{T\setminus t}^i\right). \tag{10}$$

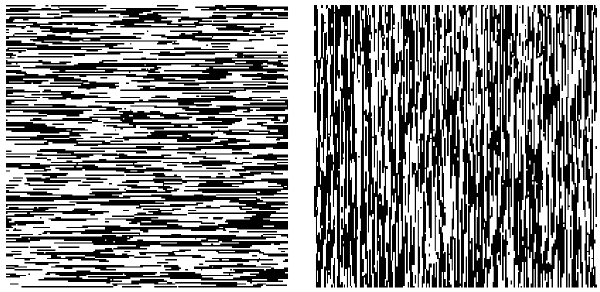
One can see from the formula (10) that when the label k_t is fixed the colors $x_t^1, x_t^2, \ldots, x_t^l$ can be generated independently. Each component of the vector $\left(x_t^1, x_t^2, \ldots, x_t^l\right)$ - x_t^i, $i \neq k_t$ is generated in accordance with the probability distribution $p\left(x_t^i | x_{T\setminus t}^i\right)$. At that the color $x_t^{k_t}$ takes the value x_t. Such an algorithm of generation of the colors $x_t^1, x_t^2, \ldots, x_t^l$ also requires $|X| \cdot l$ operations. Description of the improved generation algorithm is complete.

8. Experimental results

As it was said in Section 7, we solve the problem 3) using the generation algorithm - the solution to the problem 3*). First of all experimental verification of the segmentation algorithm is carried out on artificial examples since the true labelling of real-world images is debatable.

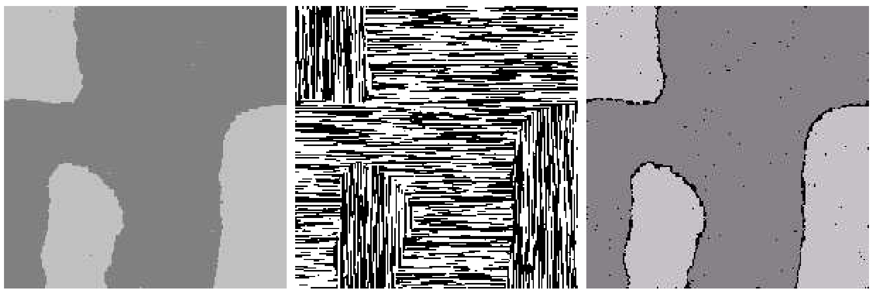
1) In the first experiment the parameters of two Markov fields defining two types of textures and the parameters of the Markov field defining the labelling were set manually. After that in accordance with the selected parameters two images (figure 2 (a), (b)) and the segmentation (figure 2 (c)) were generated. On the base of these images the input textured image x_T was constructed according to the formula (3). After that a posterior probabilities of the label in each pixel $p_t(k)$ were estimated by means of the generation method described above. In Fig. 2(e) the restored segmentation is shown. The pixels in which a posterior probability of each label is less then 0.95 are labelled by black color.

2) In the second experiment the parameters of Markov fields were obtained from the real-world texture samples using the methods described in the works [8, 9, 10, 11]. Just as in the previous example the texture images (Fig. 3(a,b,c)) and the segmentation (Fig. 3(d)) were generated. The input textured image x_T was



(a) Generated example of the first texture x_T^1

(b) Generated example of the second texture x_T^2



(c) Generated example of the labelling k_T

(d) The artificial image x_T obtained from the images (a), (b) and (c) according to the formula (3)

(e) Restored segmentation

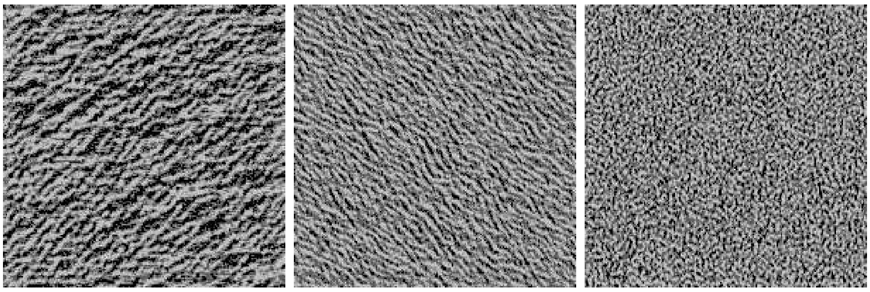
FIGURE 2. Artificial experiment with the artificial textures.

constructed from these images according to the formula (3). After that a posterior probabilities of the label in each pixel were estimated. In Fig. 3(f) the restored segmentation is shown. The pixels in which a posterior probability of each label is less then 0.95 are labelled by black color.

3) In the following experiment airphoto was chosen to be an input image for recognition. Parameters of Markov fields were obtained on the base of the texture samples shown in Fig. 4(a) using the methods described in the works [8, 9, 10, 11]. A posterior probabilities of the label in each pixel were estimated. In Fig. 4(b) the restored segmentation is shown. The pixels in which a posterior probability of each label is less then 0.95 are labelled by black color.

9. CONCLUSION

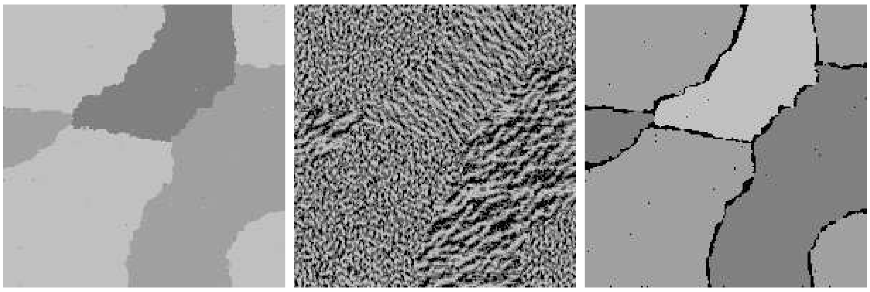
Up to now the segmentation problem was considered mainly as the maximum a posteriori probability estimation problem. Such a problem formulation disagrees with the statistical sense of the problem since it is a Bayesian problem with the



(a) Generated example of the first texture x_T^1

(b) Generated example of the second texture x_T^2

(c) Generated example of the third texture x_T^3



(d) Generated example of the labelling k_T

(e) The artificial image x_T obtained from the images (a), (b), (c) and (d) according to the formula (3)

(f) Restored segmentation

FIGURE 3. Artificial experiment with the real textures.

simplest penalty function, which presumes that the labelling that differs from the optimal one only in one pixel is as bad as the labelling that differs from the optimal one in all pixels. The additive penalty function is more appropriate in this case [7], since it expresses the natural assumption about the penalty - it depends on the number of incorrectly recognized pixels. The method for solving the Bayesian labelling estimation problem with additive function is proposed in the given owork.

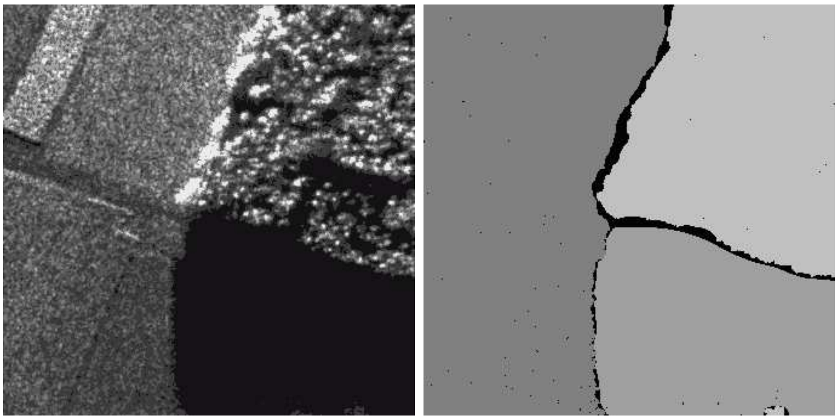
REFERENCES

[1] Stan Z. Li, "Markov Random Field Modeling in Image Analysis", Computer Science Workbench, Springer, 2001.

[2] S. Geman and D. Geman, "Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images", IEEE Trans. on PAMI, 1984, vol. 6, no. 6, pp. 721-741.

[3] V.Cerny, "A thermodynamical approach to the travelling salesman problrm: an efficient simulation algorithm", preprint, Inst. Phys. and Biophys., Comenius Univ., Bratislava, 1982.

[4] S. Kirkpatrick, C. D. Gellatt, Jr., and M. P. Vecch, "Optimization by simulated annealing", IBM Thomas J. Watson Research Center, Yorktown Heights, NY, 1982.



(a) Initial image (b) Restored segmentation

FIGURE 4. Real example with the real textures.

[5] M.I. Schlesinger , V. Hlavác, "Ten Lectures on Statistical and Structural Pattern Recognition", Kluwer Academic Publishers, Dordrecht, 2002.
[6] M.I. Schlesinger, "Mathematical facilities of the image processing", Kiev, Naukova dumka, 1989.
[7] J. Bezag, "On the statistical analysis of dirty pictures", Journal of the Royal Statistical Society, Series B 48, pp. 259-302, 1986.
[8] G. L. Gimel'farb, "Texture Modelling by Multiple Pairwise Pixel Interactions", IEEE Trans.on Pattern Anal. Machine Intell., 1996, 18:11, pp.1110-1114
[9] G. L. Gimel'farb, "Modeling image textures with Gibbs random fields", Pattern Recognition Letters, 1999, 20:12, pp.1123 - 1132.
[10] G. L. Gimel'farb, "Image Textures and Gibbs Random Fields", Kluwer Academic Publishers: Dordrecht, 1999. 250 p.
[11] Aleksey Zalesny, "Analysis and Synthesys of Textures With Pairwise Signal Interactions", Technical report: Project 50542(EC ACTS 0740-"Vanguard", 1.3.1998-1.3.1999)
[12] Alexey Zalesny and Luc Van Gool, "A Compact Model for Viewpoint Dependent Texture Synthesis", SMILE 2000 Workshop, Lecture notes in computer science, Vol. 2018, pp. 124-143, 2001.